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The coherent state and anharmonic oscillator description of nonlinear oscillation

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A perturbation solution of the nonlinear oscillation of the form

$$\ddot{x} + \omega^2 x + bx^2 + ax^3 = 0$$

is obtained, using the coherent states constructed out of quantum oscillator states. The equation of the spine obtained here is compared with that obtained by using an averaging procedure. It is found that the equation obtained in the present case is simpler and different from the other. Also the method used here has the distinct advantage that it is suitable for any nonlinear oscillator containing both even and odd higher order terms.

KEY WORDS: anharmonic oscillator, coherent states, nonlinear oscillation

1. Introduction

In many physical problems nonlinearity is an essential feature. These problems cannot be solved exactly in general and one has to take recourse to perturbation methods. In most of the cases a straightforward method breaks down as it gives rise to the so-called secular terms. To obtain information about solutions of such equations approximation or numerical approach are needed. Various methods are available for treating nonlinear oscillators analytically (see, e.g., [1–3]). The coherent state formalism [4] has the advantage that the independent variable appears in the form of $\exp(\mathrm{i}Et)$ which is a bounded function, and hence, the problem of secular terms does not arise.

A first-order solution of Duffing nonlinear oscillator (1975)

$$\ddot{x} + \omega^2 x + ax^3 = 0 \tag{1.1}$$

using the coherent states was obtained by Bhaumik and Dutta Roy [4]. Mahaffey [5] discussed some physical phenomena in plasma physics which have been found to be described by an equation in which in addition to the linear and cubic terms there was

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also the quadratic term in the nonlinear oscillator problem. The equation can be written as

$$\ddot{x} + \omega^2 x + bx^2 + ax^3 = 0. \tag{1.2}$$

At first glance, it might seem to be a trivial extension of the Duffing equation plus a constant as it can be written in the form

$$\ddot{y} + \omega_0^2 y + cy^3 + d = 0. \tag{1.3}$$

But, upon closer examination of equation (1.3) it is found that significant differences can arise between equations (1.2) and (1.3) which include additional frequency shifts in the linear frequency, effects on the symmetry of the amplitude oscillation about equilibrium, possible creation of two extra singularities in the phase plane of the hard spring case ($c > 0$), additional anharmonic structure in the resonance response of the system, and the possibility of hysteresis and jump effects on both sides of ω_0. In any physical system (neglecting dissipation) when there are oscillations about a minimum, a harmonic oscillator equation with anharmonic terms describes the oscillation. For small oscillation linear terms are sufficient, but when the oscillations are not small, higher order terms are needed.

Anharmonic oscillators are often used to test new approximation techniques since the calculation of the eigenvalues and eigenfunctions leads to challenging mathematical problems. The anharmonic oscillators are also used to test other computational approaches which are actually designed for the treatment of many-fermion system [6]. However, in this case some problems still occur [7]. Since Drinfield [8], Jimbo [9] suggested to generalize the notion of creation and annihilation operators of the usual oscillator and to introduce q oscillators [10]. Since then several attempts have been made on oscillator problems (Chaichian et al. [11]).

This observation suggests that there might exist other types of nonlinearities for which the frequency of oscillation varies with the amplitude [12]. Mancini [13] studied the behaviour of a nonlinear oscillator, plunged in a bath modelled by an assembly of harmonic oscillators, where a master equation approach was developed to consider several types of reservoir restricted to the case of small damping. Response of three degree of freedom system with cubic nonlinearities to harmonic excitation has been recently studied by El-Bassiouny and Eissa [14].

Therefore, it seems that the anharmonic oscillator defined by equation (1.2) is worth studying. In the present paper a coherent state method has been used to obtain a perturbation solution to the quantum anharmonic oscillator problem and obtain the classical solution in the appropriate limit.

Now, according to Glauber [15] the field coherent states can be constructed from any of the three mathematical definitions [16]:

(i) The coherent states $|\alpha\rangle$ are eigenstates of the harmonic oscillator annihilation operator a, that is,

$$a|\alpha\rangle = \alpha|\alpha\rangle.$$

(ii) The coherent states $|\alpha\rangle$ can be obtained by applying a displacement operator $D(\alpha)$ which is given by

$$D(a) = \exp(\alpha a^+ - \alpha^+ a).$$

(iii) The coherent states $|\alpha\rangle$ are quantum states with a minimum uncertainty relationship

$$(\Delta\overline{p})^2(\Delta\overline{q})^2 = \left(\frac{1}{2}\right)^2,$$

where the operators $\overline{q}$, $\overline{p}$ are given by

$$\overline{q} = \frac{1}{\sqrt{2}}(a + a^+), \qquad \overline{p} = \frac{1}{\sqrt{2}}(a - a^+),$$

and

$$(\Delta\overline{f})^2 = \langle\alpha|(\overline{f} - \langle\overline{f}\rangle)^2|\alpha\rangle, \quad \text{where } \langle\overline{f}\rangle = \langle\alpha|\overline{f}|\alpha\rangle.$$

We have used the definition (i) here. It can be shown that the other definitions (ii) and (iii) follow [17].

The coherent states are not orthogonal, the overlap of the states is given by

$$\left|\langle\alpha|\beta\rangle\right|^2 = \mathrm{e}^{-|\alpha-\beta|^2}.$$

$|\alpha\rangle$ and $|\beta\rangle$ are approximately orthogonal when $|\alpha - \beta|^2$ becomes large. In the next section we discuss coherent states and the classical limit.

2. The coherent states and the classical limit

The eigenstates of the Hamiltonian

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 \tag{2.1}$$

for the harmonic oscillator may be obtained easily using the annihilation and creation operators. A notation different from that used by Bhaumik and Dutta Roy [4] has been used here, for example,

$$a = \frac{m\omega x + \mathrm{i}p}{\sqrt{2\hbar m\omega}}, \qquad a^+ = \frac{m\omega x + \mathrm{i}p}{\sqrt{2\hbar m\omega}}, \tag{2.2}$$

where the commutation relations are

$$[a, a^+] = 1, \qquad [a, a] = 0 = [a^+, a^+]. \tag{2.3}$$

Some errors and misprints have been noticed in their paper which have been corrected here.

From equations (2.2) and (2.3) we get

$$H = \hbar\omega\left(a^+a + \frac{1}{2}\right). \tag{2.4}$$

The state of n quanta can be written as

$$|n\rangle = \frac{1}{\sqrt{n!}}\left(a^+\right)^n|0\rangle. \tag{2.5}$$

The coherent states for a harmonic oscillator corresponding to the energy eigenvalues $\hbar\omega(n + 1/2)$ can be obtained by superimposing these states [15]. These states are then over-complete and normalised.

One can take

$$|\alpha\rangle = \exp\left(\frac{-\alpha^2}{2}\right)\sum_0^\infty \frac{\alpha^n}{\sqrt{n!}}|n\rangle, \tag{2.6}$$

where α is a complex number. These numbers are eigenvalues of the annihilation operator a, that is,

$$a|\alpha\rangle = \alpha|\alpha\rangle. \tag{2.7}$$

It can be easily shown that

$$\langle\alpha|x|\alpha\rangle = 2\lambda\sqrt{\frac{\hbar}{2m\omega}}\cos\omega t, \tag{2.8}$$

where

$$\alpha = -\mathrm{i}\lambda\exp(\mathrm{i}\omega t). \tag{2.9}$$

Hence, for the harmonic oscillator case one can take

$$|\alpha\rangle = N_\alpha\sum_{n=0}^\infty \frac{(-\mathrm{i}\lambda)^n}{\sqrt{n!}}\exp\left(\frac{\mathrm{i}E_n t}{\hbar}\right)|n\rangle, \tag{2.10}$$

where we have written E_n for $n\omega$ and N_α is the normalization constant.

The classical limit is obtained by letting $\hbar \to 0$, $\lambda \to \infty$ with

$$2\lambda\sqrt{\frac{\hbar}{2m\omega}} \to A,$$

A being the corresponding classical amplitude of the classical oscillator.

3. The anharmonic oscillator and coherent states

The Hamiltonian for the nonlinear oscillator described by equation (1.2) is

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2x^2 + \frac{1}{3}bx^3 + \frac{1}{4}ax^4. \tag{3.1}$$

The coherent state method has been applied to this perturbed Hamiltonian to obtain the eigenstates $|\alpha\rangle'$ in terms of the modified eigenstates $|n\rangle'$.

To get the coherent state for the Hamiltonian (3.1) we replace E_n by E'_n in equation (2.10), E'_n being the perturbed energy for the anharmonic oscillator (3.1), and we also replace $|n\rangle$ by the perturbed state $|n\rangle'$. Both E'_n and $|n\rangle'$ are well known. Hence, we write

$$|\alpha\rangle = N_\lambda \sum_{n=0}^{\infty} \frac{(-\mathrm{i}\lambda)^n}{\sqrt{n!}} \exp\left(\frac{\mathrm{i}E'_n t}{\hbar}\right)|n\rangle', \tag{3.2}$$

N_λ being the normalization constant.

The perturbed state is given by

$$\begin{aligned}|n\rangle' = |n\rangle &+ \frac{ma}{4\hbar\omega}\left(\frac{\hbar}{2m\omega}\right)^2\left[-\frac{1}{4}\sqrt{(n+1)(n+2)(n+3)(n+4)}|n+4\rangle\right.\\ &+ (2n+3)\sqrt{(n+1)(n+2)}|n+2\rangle - (2n-1)\sqrt{n(n-1)}|n-2\rangle\\ &\left.+ \frac{1}{4}\sqrt{n(n-1)(n-2)(n-3)}|n-4\rangle\right]\\ &+ \frac{mb}{3\hbar\omega}\left(\frac{\hbar}{2m\omega}\right)^{3/2}\left[-\frac{1}{3}\sqrt{(n+1)(n+2)(n+3)}|n+3\rangle\right.\\ &- 3(n+1)\sqrt{(n+1)}|n+1\rangle + 3n\sqrt{n}|n-1\rangle\\ &\left.+ \frac{1}{3}\sqrt{n(n-1)(n-2)}|n-3\rangle\right] + \mathrm{O}\big(a^2\big) + \mathrm{O}\big(b^2\big),\end{aligned} \tag{3.3}$$

whereas the perturbation energy is given by

$$E'_n = n\hbar\omega + \frac{\hbar^2}{4m\omega^2}\left\{\frac{3a}{2}\left(n^2+n+\frac{1}{2}\right) - \frac{5b^2}{3\omega^2}\left(n^2+n+\frac{11}{20}\right)\right\}. \tag{3.4}$$

In view of the Ehrenfest theorem [18] and the vanishing quantum correlations in the classical limit, the solution of the classical problem can be obtained from the classical limit of the quantum solution. Thus, to first order in a and b

$$\begin{aligned}\langle x\rangle = \lim_{\lambda\to\infty}\langle\alpha'|x|\alpha'\rangle &= A\cos\omega_1 t - \frac{bA^2}{6\omega_1^2}(\cos 2\omega_1 t + 3)\\ &+ \frac{aA^3}{32\omega_1^2}(\cos 3\omega_1 t - 6\cos\omega_1 t) + \mathrm{O}\big(a^2\big) + \mathrm{O}\big(b^2\big),\end{aligned} \tag{3.5}$$

where

$$\omega_1 = \omega\left(1 + \frac{3}{8\omega^2}aA^2 - \frac{5}{12}\frac{b^2A^2}{\omega^4}\right). \tag{3.6}$$

4. Conclusions and discussion

Spines (spines are essentially the curves showing relation between the amplitude and frequency [5]) of the resonance response curve can be obtained by starting with equation (1.2). Mahaffey [5] assumed a solution of the equation (1.2) in the form (using the averaging procedure)

$$\bar{x} = A\cos\omega_1 t + C$$

and found that ω_1 satisfies the equation

$$\left(\frac{9a^2}{4}\right)A^4 + \left\{3a\left(\frac{3\omega^2}{2} - \omega_1^2\right) - 2b^2\right\}A^2 + 2\omega^2\left(\omega^2 - \omega_1^2\right) = 0, \tag{4.1}$$

where

$$C = -bA^2\left[2\left(\omega^2 + \frac{3A^2a}{2}\right)\right]^{-1}. \tag{4.2}$$

The value of C obtained from (4.2) is the same as that obtained here (see equation (3.5)). However, the equation for spine as shown in (4.1) is different from the result shown in equation (3.6). In the present paper x is an operator and we can only compare $\langle\alpha'|x|\alpha'\rangle$ with Mahaffey's solution. The result obtained here is less complicated than that of Mahaffey [5]. One significant difference is that in the result obtained here, there is no real root for $b^2 > 3a\omega^2/4$.

However, owing to the presence of the b term the other conclusions reached in the paper by Bhaumik and Dutta Roy [4] still remain valid, for example, the asymmetry of the average value of frequency taken over the half-cycle and the asymmetry of

$$\left(\frac{\mathrm{d}\overline{A}}{\mathrm{d}t}\right)_{0,\pi/2} \quad \text{and} \quad \left(\frac{\mathrm{d}\overline{A}}{\mathrm{d}A}\right)_{\pi,3\pi/2}.$$

These can be easily verified by putting the solution (3.3) in equation (1.2) and taking averages of $\omega(t)$ and $A(t)$. Recently, quartic, sextic and octic anharmonic oscillators were studied by Meibner et al. [2,3] using WEC – iteration method which produces accurate approximations to the energies of x^{2m} anharmonic oscillators ($m = 2, 3, 4, \ldots$). But they did not include any odd powers of x. Though we have considered the anharmonic oscillator with cubic and fourth-order terms here, the method could be extended to any nonlinear oscillator containing higher order terms of both even and odd orders.

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